

Extended Einstein-Cartan theory à la Diakonov: the field equations

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Abstract

Diakonov formulated a model of a primordial Dirac spinor field interacting gravitationally within the geometric framework of the Poincaré gauge theory (PGT). Thus, the gravitational field variables are the orthonormal coframe (tetrad) and the Lorentz connection. A simple gravitational gauge Lagrangian is the Einstein-Cartan choice proportional to the curvature scalar plus a cosmological term. In Diakonov's model the coframe is eliminated by expressing it in terms of the primordial spinor. We derive the corresponding field equations for the first time. We extend the Diakonov model by additionally *eliminating the Lorentz connection*, but keeping local Lorentz covariance intact. Then, if we drop the Einstein-Cartan term in the Lagrangian, a nonlinear Heisenberg type spinor equation is recovered in the lowest approximation.

Keywords: Poincaré gauge theory, Diakonov model, Einstein–Cartan theory, Heisenberg spinor equation *file dia11.tex, 27 Feb 2012*

1. Introduction

Recently Diakonov [1] developed a scheme for the quantization of the gravitational field interacting with a primordial *Dirac spinor* field ψ . His starting point is the framework of the Poincaré gauge theory of gravity

(PGT); therein gravity is described by two independent fields, by the *orthonormal coframe* $\vartheta^\alpha = e_i^\alpha dx^i$ (translational potential) and by the *Lorentz connection* $\Gamma^{\alpha\beta} = \Gamma_i^{\alpha\beta} dx^i = -\Gamma^{\beta\alpha}$ (Lorentz potential); for references see [2, 3, 4], our *notation* is explained at the end of this introduction. Accordingly, the original field variables of Diakonov's theory are $(\vartheta^\alpha, \Gamma^{\alpha\beta}, \psi, \bar{\psi})$.

In the Appendix we displayed the most general gravitational Lagrangian of PGT that is *quadratic* in the torsion T^α and in the curvature $R^{\alpha\beta} = -R^{\beta\alpha}$ and encompasses all parity even and odd pieces [5, 6, 7, 8]. At first Diakonov considers only the simplest model, namely the first two pieces on the right-hand-side of Eq. (44). They constitute the gravitational Lagrangian of the Einstein-Cartan(-Sciama-Kibble) theory (ECT), see [9]:

$$V = -\frac{1}{2\kappa} \left(\eta_{\alpha\beta} \wedge R^{\alpha\beta} - 2\Lambda\eta \right). \quad (1)$$

Here Λ is the cosmological constant. The matter part is typically the minimally coupled Dirac Lagrangian 4-form

$$\begin{aligned} L_D &= \frac{i}{2} \left(\bar{\psi} \star \gamma \wedge D\psi + \overline{D\psi} \wedge \star \gamma \psi \right) + \star mc \bar{\psi} \psi \\ &= -\left[\frac{i}{2} e_\alpha \right] \underbrace{\left(\bar{\psi} \gamma^\alpha D\psi - D\bar{\psi} \gamma^\alpha \psi \right)}_{1\text{-form}} - mc \bar{\psi} \psi \Big] \eta. \end{aligned} \quad (2)$$

For later use we isolated in the second line the volume 4-form η .

The Einstein-Cartan-Dirac theory of gravity has the total Lagrangian $L = V + L_D$, with the action $W = \int L$. The two gravitational field equations read

$$\frac{1}{2\kappa} R^{\rho\sigma} \wedge \eta_{\alpha\rho\sigma} - \frac{\Lambda}{\kappa} \eta_\alpha = \Sigma_\alpha^D = \frac{i}{2} \left(\bar{\psi} \star \gamma D_\alpha \psi - D_\alpha \bar{\psi} \star \gamma \psi \right), \quad (3)$$

$$\frac{1}{2\kappa} T^\rho \wedge \eta_{\alpha\beta\rho} = \tau_{\alpha\beta}^D = -\frac{1}{8} \bar{\psi} (\star \gamma \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \star \gamma) \psi. \quad (4)$$

As sources of the field equations act the canonical 3-forms of energy-momentum Σ_α^D and spin angular momentum and $\tau_{\alpha\beta}^D$ of the Dirac field, respectively, see [10]. The second field equation (4) is algebraic in the torsion and can be resolved with respect to the *contortion* 1-form $K_\alpha^\beta = \tilde{\Gamma}_\alpha^\beta - \Gamma_\alpha^\beta$, where $\tilde{\Gamma}_\alpha^\beta(\vartheta^\vartheta, d\vartheta^\alpha)$ is the Riemannian piece of the connection and $T^\alpha = K^\alpha_\beta \wedge \vartheta^\beta$:

$$K^{\alpha\beta} = -\kappa \star \tau^{D\alpha\beta} = \frac{\kappa}{4} \eta^{\alpha\beta\rho} \bar{\psi} \gamma_\rho \gamma_5 \psi = \frac{\kappa}{4} \star (\vartheta^{\alpha\beta} \wedge \bar{\psi} \gamma \gamma_5 \psi). \quad (5)$$

In vacuum, where the Dirac field vanishes, contortion and torsion vanish, too, and we recover the vacuum Einstein theory of gravity.

Notation [3, 4]: Latin letters $i, j, k, \dots = 0, 1, 2, 3$ denote (holonomic) coordinate indices, Greek letters $\alpha, \beta, \gamma, \dots = \hat{0}, \hat{1}, \hat{2}, \hat{3}$ (anholonomic) frame indices. Symmetrization over indices is denoted by $(\alpha\beta) := \{\alpha\beta + \beta\alpha\}/2$, antisymmetrization by $[\alpha\beta] := \{\alpha\beta - \beta\alpha\}/2$. The frame $e_\beta = e^j_\beta \partial_j$ is dual to the coframe: $e_\beta \lrcorner \vartheta^\alpha = \delta_\beta^\alpha$. Greek indices are raised and lowered by means of the Minkowski metric $g_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$. The volume 4-form is denoted by $\eta = \star 1$, and $\eta_\alpha = \star \vartheta_\alpha$, $\eta_{\alpha\beta} = \star \vartheta_{\alpha\beta}$, $\eta_{\alpha\beta\gamma} = \star \vartheta_{\alpha\beta\gamma}$, $\eta_{\alpha\beta\gamma\delta} = \star \vartheta_{\alpha\beta\gamma\delta}$, where \star is the Hodge star operator and $\vartheta^{\alpha\beta} := \vartheta^\alpha \wedge \vartheta^\beta$, etc. Furthermore, $T^\alpha := D\vartheta^\alpha = T_{ij}^\alpha dx^i \wedge dx^j / 2$, $T_{ij}^\alpha = 2(\partial_{[i}\vartheta_{j]}^\alpha + \Gamma_{[ij]}^\alpha)$, $R^{\alpha\beta} := d\Gamma^{\alpha\beta} + \Gamma^{\alpha\gamma} \wedge \Gamma^\beta_\gamma = R_{ij}^{\alpha\beta} dx^i \wedge dx^j / 2$, $R_{ij}^{\alpha\beta} = 2(\partial_{[i}\Gamma_{j]}^{\alpha\beta} + \Gamma_{[i}^{\alpha\gamma} \Gamma_{j]}^{\beta\gamma})$, $\text{Ric}_\alpha := e_\beta \lrcorner R^{\beta\alpha} = \text{Ric}_{\beta\alpha} \vartheta^\beta$, with $\text{Ric}_{\alpha\beta} = R_{\gamma\alpha\beta}{}^\gamma$, $R := \text{Ric}_\alpha{}^\alpha = R_{\beta\alpha}{}^{\alpha\beta}$. Our master formula for the variation of coframe and metric is [11]:

$$(\delta^\star - \star \delta) \phi = \delta \vartheta^\alpha \wedge (e_\alpha \lrcorner \star \phi) - \star [\delta \vartheta^\alpha \wedge (e_\alpha \lrcorner \phi)] + \delta g_{\alpha\beta} [\vartheta^{(\alpha} \wedge (e^{\beta)} \lrcorner \star \phi) - \frac{1}{2} g^{\alpha\beta} \star \phi]. \quad (6)$$

In the Dirac theory we have the 1-form $\gamma := \gamma_\alpha \vartheta^\alpha$ and for the Dirac adjoint $\bar{\psi} := \psi^\dagger \gamma^0$; the exterior spinor covariant derivative is given by $D = d + i\sigma_{\alpha\beta} \Gamma^{\alpha\beta} / 4$ and the Lorentz algebra generators by $\sigma^{\alpha\beta} = i\gamma^{[\alpha} \gamma^{\beta]}$. Moreover, we put $\hbar = 1, c = 1$.

2. Diakonov's model

Diakonov [1] proposed recently a model of “microscopic quantum gravity” such that the coframe ϑ^α is dropped as an independent field and expressed in terms of a primordial anticommuting Dirac field Ψ , that is $\vartheta^\alpha = \vartheta^\alpha(\Psi, \bar{\Psi})$. Following similar ideas of Akama [12], he made the ansatz

$$\ell^{-4} \vartheta^\alpha = \frac{i}{2} (\bar{\Psi} \gamma^\alpha D\Psi - D\bar{\Psi} \gamma^\alpha \Psi) \quad (7)$$

or $\varphi^\alpha = 0$, with

$$\varphi^\alpha(\vartheta^a, \Gamma^{\alpha\beta}, \Psi, \bar{\Psi}, d\Psi, d\bar{\Psi}) := \frac{i}{2} (\bar{\Psi} \gamma^\alpha D\Psi - D\bar{\Psi} \gamma^\alpha \Psi) - \ell^{-4} \vartheta^\alpha. \quad (8)$$

The constant ℓ carries the dimension of a length. Apparently, the primordial spinor is assumed to be massless. In contrast to ϑ^α , the connection $\Gamma^{\alpha\beta}$ is upheld as an independent field.

We recognize on the right-hand-side of (7) the 1-form marked in (2). A comparison with Σ_α^D in (3) and some algebra yields an alternative form of (7). We first define the *transposed* of an arbitrary covector-valued 1-form $E_\alpha = E_{i\alpha} dx^i$ as $\widehat{E}_\alpha := E_\alpha + e_\alpha \lrcorner (E_\beta \wedge \vartheta^\beta) = (e^\alpha \lrcorner E_\beta) \vartheta^\beta$. It is simple to

verify that $\widehat{E_{i\alpha}} = E_{\alpha i}$. With this new transposition operator we can write Diakonov's formula (7) succinctly as

$$\Sigma_\alpha^D = \ell^{-4} g_{\alpha\beta} \star \widehat{\vartheta^\beta} \quad (9)$$

This is reminiscent of the Hooke's type constitutive law in Cosserat elasticity, see [13]: the (asymmetric) force stress 3-form Σ^D is proportional to the distortion 1-form ϑ , with $\ell^{-4} g_{\alpha\beta}$ as the elasticity modulus.¹

Diakonov [1] assumes that the primordial spinor matter acts only via the coframe ϑ^α according to (7), (8), or (9). Thus, no explicit matter part enters the Lagrangian. Most conveniently we stick with the original field variables $(\vartheta^\alpha, \Gamma^{\alpha\beta}, \Psi, \overline{\Psi})$ and add the constraint $\varphi^\alpha = 0$ with suitable Lagrange multiplier 3-forms λ_α to the EC-Lagrangian V . In this way the local Poincaré covariance of the PGT is upheld. Thus, the *Diakonov Lagrangian* reads

$$L = V + L_{\text{mat}} = -\frac{1}{2\kappa} \left(a_0 R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\Lambda \eta \right) + L_{\text{mat}}, \quad (10)$$

with the matter Lagrangian

$$L_{\text{mat}} = L(\vartheta^\alpha, \Gamma^{\alpha\beta}, \Psi, \overline{\Psi}, d\Psi, d\overline{\Psi}, \lambda_\alpha) := \varphi^\alpha \wedge \lambda_\alpha. \quad (11)$$

The dimensionless constant parameter a_0 is introduced for generality, compare (44). We can immediately calculate the translation and Lorentz *excitations*:

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha} = 0, \quad H_{\alpha\beta} := -\frac{\partial V}{\partial R^{\alpha\beta}} = \frac{a_0}{2\kappa} \eta_{\alpha\beta}. \quad (12)$$

In turn, we find for the energy-momentum and the spin angular momentum of the *gauge fields*

$$\begin{aligned} E_\alpha &:= e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R^{\rho\sigma}) \wedge H_{\rho\sigma} \\ &= -\frac{a_0}{2\kappa} R^{\rho\sigma} \wedge \eta_{\alpha\rho\sigma} + \frac{\Lambda}{\kappa} \eta_\alpha, \end{aligned} \quad (13)$$

$$E_{\alpha\beta} := -\vartheta_{[\alpha} \wedge H_{\beta]} = 0. \quad (14)$$

¹This similarity would be even more pronounced, if one dropped the transposition operator and took the gauge-theoretically more satisfactory ansatz $\Sigma_\alpha^D = \ell^{-4} \star \vartheta_\alpha$.

Concentrating now on the matter Lagrangian (11), we find for the material energy-momentum and spin currents, respectively,

$$\Sigma_\alpha := \frac{\delta L_{\text{mat}}}{\delta \vartheta^a} = -\ell^{-4} \lambda_\alpha, \quad (15)$$

$$\tau_{\alpha\beta} := \frac{\delta L_{\text{mat}}}{\delta \Gamma^{\alpha\beta}} = -\frac{1}{8} \bar{\Psi}(\gamma^\rho \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \gamma^\rho) \Psi \lambda_\rho, \quad (16)$$

3. Field equations

The two classical field equations of gravity read

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \quad (17)$$

$$DH_{\alpha\beta} - E_{\alpha\beta} = \tau_{\alpha\beta}. \quad (18)$$

Consequently, the *first* field equation (17), with the help of (12)₁ and (13), determines the Lagrange multiplier

$$\ell^{-4} \lambda_\alpha = -\frac{a_0}{2\kappa} R^{\rho\sigma} \wedge \eta_{\alpha\rho\sigma} + \frac{\Lambda}{\kappa} \eta_\alpha, \quad (19)$$

whereas the *second* field equation (18), by means of (12)₂ and (14), reduces to a nontrivial equation for the torsion:

$$\frac{a_0}{2\kappa} T^\rho \wedge \eta_{\alpha\beta\rho} = -\frac{1}{8} \bar{\Psi}(\gamma^\rho \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \gamma^\rho) \Psi \lambda_\rho. \quad (20)$$

If we substitute the Lagrange multiplier λ_α into the right-hand-side of (20), we recognize that, in contrast to the ECT, in Diakonov's theory the *torsion is non-trivial*.

Eventually, variation of (10) with respect to the primordial spinor yields the *generalized Dirac* equation

$$i\gamma^\alpha \lambda_\alpha \wedge D\Psi - \frac{i}{2} \gamma^\alpha (D\lambda_\alpha) \Psi = 0. \quad (21)$$

Using (19), we have for the derivative of the Lagrange multiplier

$$D\lambda_\alpha = -\frac{\ell^4}{\kappa} \left(\frac{a_0}{2} \eta_{\alpha\beta\rho\sigma} R^{\rho\sigma} - \Lambda \eta_{\alpha\beta} \right) \wedge T^\beta. \quad (22)$$

Variation with respect to the Lagrange multiplier 3-form λ_α yields the *constraint*

$$\varphi^\alpha = 0 \quad (23)$$

which reproduces (7) or the “constitutive law” (9).

4. A possible generalization of Diakonov's model?

Since Diakonov wanted to keep the local Lorentz invariance of the action of his model, he did not express the Lorentz connection in terms of the primordial spinor. But there is no convincing rationale behind this argument. With our interpretation (9) of the Akama–Diakonov ansatz (7), the generalization to the elimination of the Lorentz connection is straightforward. In Cosserat elasticity, the second constitutive law relates the spin moment stress 3-form (torque) τ^D linearly to the rotational deformation measure, the contortion 1-form K (see [13], Sec. 3):

$$\tau_{\alpha\beta}^D = L^{-2} g_{\alpha\gamma} g_{\beta\delta} {}^\star K^{\gamma\delta} = L^{-2} {}^\star (\tilde{\Gamma}_{\alpha\beta}(\vartheta, d\vartheta) - \Gamma_{\alpha\beta}) . \quad (24)$$

Here the constant L of the “rotational modulus” $L^{-2} g_{\alpha\gamma} g_{\beta\delta}$ carries the dimension of a length. As new constraint we have then $\varphi^{\alpha\beta} = 0$, with

$$\varphi^{\alpha\beta}(\vartheta^a, d\vartheta^a, \Gamma^{\alpha\beta}, \Psi, \bar{\Psi}) := \frac{1}{4} {}^\star (\vartheta^{\alpha\beta} \wedge \bar{\Psi} \gamma_5 \Psi) - L^{-2} (\tilde{\Gamma}^{\alpha\beta}(\vartheta, d\vartheta) - \Gamma^{\alpha\beta}) \quad (25)$$

[see [10], Eq. (22)]. As a result, the matter Lagrangian (11) is generalized to

$$L_{\text{mat}} = L(\vartheta^\alpha, d\vartheta^\alpha, \Gamma^{\alpha\beta}, \Psi, \bar{\Psi}, d\Psi, d\bar{\Psi}, \lambda_\alpha, \lambda_{\alpha\beta}) = \varphi^\alpha \wedge \lambda_\alpha + \varphi^{\alpha\beta} \wedge \lambda_{\alpha\beta}, \quad (26)$$

where we introduced a second Lagrange multiplier 3-form $\lambda_{\alpha\beta} = -\lambda_{\beta\alpha}$.

We could now execute the variational calculations explicitly.² However, there is a simpler method applicable. We consider the Lagrange multiplier as an additional matter field and employ the Lagrange-Noether machinery as described in [3], Sec. 5, for example. The constraints $\varphi^\alpha = 0$ and $\varphi^{\alpha\beta} = 0$ are used thereby. After some algebra, we find that for the matter Lagrangian (26) the formulas (15) and (16) are generalized as follows:

$$\Sigma_\alpha := \frac{\delta L_{\text{mat}}}{\delta \vartheta^a} = -\ell^{-4} \lambda_\alpha + L^{-2} \widehat{D} \xi_\alpha, \quad (27)$$

$$\tau_{\alpha\beta} := \frac{\delta L_{\text{mat}}}{\delta \Gamma^{\alpha\beta}} = -\frac{1}{8} \bar{\Psi} (\gamma^\rho \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \gamma^\rho) \Psi \lambda_\rho + L^{-2} \lambda_{\alpha\beta}. \quad (28)$$

²Direct variation is spelled out in the following formula:

$$\Sigma_\alpha = -\ell^{-4} \lambda_\alpha + \left(\delta \left[{}^\star (\vartheta^{\mu\nu} \wedge \bar{\Psi} \gamma_5 \Psi) / 4 - L^{-2} \tilde{\Gamma}^{\mu\nu}(\vartheta, d\vartheta) \right] / \delta \vartheta^\alpha \right) \wedge \lambda_{\mu\nu} .$$

In this case, the master formula (6) has to be applied repeatedly. In a similar way, we can also compute $\tau_{\alpha\beta}$.

Here we introduced the covector-valued 2-form ξ_α which is equivalent to the antisymmetric tensor-valued Lagrange multiplier 3-form $\lambda_{\alpha\beta}$:

$$\xi_\alpha = L^2 \frac{\partial L_{\text{mat}}}{\partial T^\alpha} = 2e^\rho \rfloor \lambda_{\rho\alpha} + \frac{1}{2} \vartheta_\alpha \wedge (e^\rho \rfloor e^\sigma \rfloor \lambda_{\rho\sigma}); \quad (29)$$

both fields have 24 independent components (see [3], Eq. (5.1.24)). Furthermore, $\widehat{D} \xi_\alpha$ is the covariant exterior derivative with respect to the transposed connection $\widehat{\Gamma}_\alpha{}^\beta := \Gamma_\alpha{}^\beta - e_\alpha \rfloor T^\beta$, see [3], Eq. (3.11.9).

Let us collect our results. We have a Lagrangian $L = V + L_{\text{mat}}$, with the matter Lagrangian (26). In the framework of the Poincaré gauge theory (PGT), V is the quadratic gauge Lagrangian (44). The general field equations are (17) and (18). If we substitute the excitations $(H_\alpha, H_{\alpha\beta})$, see (12), and the gauge currents $(E_\alpha, E_{\alpha\beta})$, see (13) and (14), respectively, we find

$$-D \frac{\partial V}{\partial T^\alpha} - e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge \frac{\partial V}{\partial T^\beta} + (e_\alpha \rfloor R^{\beta\gamma}) \wedge \frac{\partial V}{\partial R^{\beta\gamma}} = \Sigma_\alpha, \quad (30)$$

$$-D \frac{\partial V}{\partial R^{\alpha\beta}} + \vartheta_{[\alpha} \wedge \frac{\partial V}{\partial T^{\beta]}} = \tau_{\alpha\beta}. \quad (31)$$

The sources on the right-hand-sides of these gauge field equations are provided by (27) and (28). In other words, Eqs. (30) and (31) merely determine the Lagrange multipliers λ_α and $\lambda_{\alpha\beta}$; and they do it for the complete quadratic Lagrangian displayed in (44). In other words, we have eliminated by this procedure coframe and Lorentz connection altogether within the framework of the quadratic PGT.

For the general Lagrangian (44), these field equations are very complicated. For the Diakonov Lagrangian with the Einstein-Cartan term plus cosmological constant, see (10), they reduce in our generalized framework to

$$\frac{a_0}{2\kappa} R^{\rho\sigma} \wedge \eta_{\alpha\rho\sigma} - \frac{\Lambda}{\kappa} \eta_\alpha = -\ell^{-4} \lambda_\alpha + L^{-2} \widehat{D} \xi_\alpha, \quad (32)$$

$$\frac{a_0}{2\kappa} T^\rho \wedge \eta_{\alpha\beta\rho} = -\frac{1}{8} \overline{\Psi} (\gamma^\rho \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \gamma^\rho) \Psi \lambda_\rho + L^{-2} \lambda_{\alpha\beta}. \quad (33)$$

Note that for $L \rightarrow \infty$, we *recover* the Diakonov model, see (19) and (20). The dynamics of the theory is contained in the nonlinear spinor equation that we obtain by varying the action with respect to the spinor field:

$$i\gamma^\alpha \lambda_\alpha \wedge D\Psi - \frac{i}{2} \gamma^\alpha (D\lambda_\alpha) \Psi + \frac{1}{4} \vartheta^{\alpha\beta} \wedge \star \lambda_{\alpha\beta} \wedge \gamma \gamma_5 \Psi = 0. \quad (34)$$

Accordingly, our generalized Diakonov model, in which eventually only the primordial spinor is left as field variable, is controlled by the field equations (32), (33), (34), $\varphi^\alpha = 0$, $\varphi^{\alpha\beta} = 0$.

As already observed by Akama [12] and Diakonov [1], the absolute simplest model would be to *drop the Einstein-Cartan Lagrangian* by putting a_0 to zero. This reduces the gravitational Lagrangian to the cosmological term, that is, just to a volume integral:

$$L = \frac{\Lambda}{\kappa} \eta + \varphi^\alpha \wedge \lambda_\alpha + \varphi^{\alpha\beta} \wedge \lambda_{\alpha\beta}. \quad (35)$$

Then the field equations (32), (33), with $a_0 = 0$, determine the Lagrange multipliers according to

$$\ell^{-4} \lambda_\alpha = \frac{\Lambda}{\kappa} \eta_\alpha + \frac{1}{L^2} \widehat{D} \xi_\alpha, \quad (36)$$

$$L^{-2} \lambda_{\alpha\beta} = \frac{1}{8} \overline{\Psi} (\gamma^\rho \sigma_{\alpha\beta} + \sigma_{\alpha\beta} \gamma^\rho) \Psi \lambda_\rho. \quad (37)$$

In general, the differential-algebraic system (36), (37), (34) is quite non-trivial. One can try to *solve it iteratively*, then in the first approximation we find

$$\lambda_\alpha = \frac{\Lambda \ell^4}{\kappa} \eta_\alpha, \quad {}^* \lambda_{\alpha\beta} = - \frac{\Lambda \ell^4 L^2}{4\kappa} \eta_{\alpha\beta\rho\sigma} \vartheta^\rho \overline{\Psi} \gamma^\sigma \gamma_5 \Psi. \quad (38)$$

As a consequence (34) reduces to

$${}^* \gamma \wedge iD\Psi - \frac{3}{8} L^2 {}^* (\overline{\Psi} \gamma \gamma_5 \Psi) \wedge \gamma \gamma_5 \Psi = 0. \quad (39)$$

In this zeroth approximation, the gravitational constant $\kappa = \ell_{\text{Pl}}^2$, the cosmological constant $\Lambda = 1/\ell_{\text{cos}}^2$, and the constant ℓ all drop out. In the Einstein-Cartan theory we have a critical length $\ell_{\text{EC}} = (\lambda_{\text{C}} \ell_{\text{Pl}}^2)^{1/3}$, where λ_{C} is the Compton wavelength typically of the nucleon. The problem of these different length scales was not discussed by Diakonov. One guess would be that L in (39) has to be identified with ℓ_{EC} , but the situation is far from clear to us. In any case, the dimensionless constant $\beta := (\Lambda \ell^4 / \kappa)^{1/2} = \ell^2 / (\ell_{\text{Pl}} \ell_{\text{cos}})$ in (38)₁, that is, in $\lambda_\alpha = \beta^2 \eta_\alpha$, possibly plays an important role.

Thus, in the lowest approximation we recover a nonlinear spinor equation of the Heisenberg-Pauli type [14], which was once part of one of the most advanced models [17] that attempted to describe all physical interactions in

terms of a fundamental fermion field, see also Ivanenko's work on nonlinear spinor equations [18]. Kibble [15] and Rodichev [16] were the first to recognize that such a type of equation emerges in the context of the Dirac theory automatically provided spacetime carries *Cartan's torsion* in the context of a gravitational gauge theory.

5. Discussion

Let us consider a special case of our generalized model that is in a sense *complementary* to the model of Diakonov. Namely, we impose only the second constraint (25) for the connection, but forget about the original constraint (8) for the coframe. Then the gravitational field equations (17), (18) reduce to

$$\frac{a_0}{2\kappa} R^{\rho\sigma} \wedge \eta_{\alpha\rho\sigma} + \frac{\Lambda}{\kappa} \eta_\alpha = \frac{a_0}{2\kappa} [T_\alpha \wedge \bar{\Psi} \gamma \gamma_5 \Psi + \vartheta_\alpha \wedge D(\bar{\Psi} \gamma \gamma_5 \Psi)], \quad (40)$$

$$L^{-2} \lambda_{\alpha\beta} = \frac{a_0}{2\kappa} T^\rho \wedge \eta_{\alpha\beta\rho} = -\frac{a_0}{4\kappa} \vartheta_{\alpha\beta} \wedge \bar{\Psi} \gamma \gamma_5 \Psi. \quad (41)$$

We thus see that in the complementary picture with only a connection constraint, the coframe is determined from the Einstein like gravitational field equation (40) with some effective energy-momentum current constructed from the spinor fields. The second field equation (41) determines the Lagrange multiplier $\lambda_{\alpha\beta}$. This is very different from the original Diakonov model with the coframe constraint. The first equation (19) determines the Lagrange multiplier λ_α , whereas the second equation (20) describes the non-trivial spacetime torsion.

In this sense, the Diakonov theory can be called a “Cartan-connection” model, and the complementary theory an “Einstein-coframe” (or “-tetrad”) model. Diakonov uses a Hooke's type law as constraint

$$\Sigma_\alpha^D = \ell^{-4} g_{\alpha\beta} \widehat{\star \vartheta^\beta}, \quad (42)$$

in the complementary case a MacCullagh's type law, see [19], for the rotationally elastic aether is employed (see its modern incarnation in [20] and compare [21, 22]),

$$\tau_{\alpha\beta}^D = L^{-2} g_{\alpha\gamma} g_{\beta\delta} \star K^{\gamma\delta}. \quad (43)$$

In our extended Diakonov model we postulate the constitutive equations of an elastic Cosserat continuum, which is responsive to translational and

rotational deformations, that is, we take both, Hooke's and MacCullagh's law at the same time. Such pictures from continuum mechanics (classical field theory) helped also Hammad [23] to find an entropy functional for a graviatational theory acting in a Riemann-Cartan spacetime.

The extended Diakonov model appears to be a generalization of the old Heisenberg nonlinear spinor theory that is recovered in the lowest approximation in equation (39). The quantization of the full theory is a difficult task, and it will not be discussed here. Diakonov [1] is using a lattice approach and work is in progress.

Despite the fact that the ansatz for the coframe (7) was inspired by the work of Akama [12], the model of Akama is only invariant under the global Lorentz group. In contrast, the Diakonov [1] and the extended Diakonov models are explicitly invariant under *local* Lorentz transformations. This is a result of the consistent use of the geometrical framework of the Poincaré gauge theory of gravity.

Appendix: Most general quadratic Lagrangian in PGT

The PGT with its two field equations (17) and (18) is only complete if we specify its gauge Lagrangian. As a typical gauge theory, the Lagrangian is quadratic in the field strengths torsion $T^\alpha = \sum_{I=1,2,3} {}^{(I)}T^\alpha$ and curvature $R^{\alpha\beta} = \sum_{K=1,2,\dots,6} {}^{(K)}R^{\alpha\beta}$; here torsion and curvature are represented as sums in terms of their irreducible pieces.

If we introduce the notations R and X for the curvature scalar and the curvature pseudoscalar, then we find ${}^{(6)}R_{\alpha\beta} = -R\vartheta_{\alpha\beta}/12$ and ${}^{(3)}R_{\alpha\beta} = -X\eta_{\alpha\beta}/12$, respectively; moreover, for the torsion we can define the 1-forms of \mathcal{A} and \mathcal{V} for the axial vector and the vector torsion ${}^{(3)}T^\alpha = \star(\mathcal{A} \wedge \vartheta^\alpha)/3$ and ${}^{(2)}T^\alpha = -(\mathcal{V} \wedge \vartheta^\alpha)/3$, respectively. Our final gravitational Lagrangian is then [5, 6, 7, 8]

$$\begin{aligned}
V = & -\frac{1}{2\kappa} \left[(a_0 R - 2\Lambda + b_0 X) \eta \right. \\
& + \frac{a_2}{3} \mathcal{V} \wedge \star \mathcal{V} - \frac{a_3}{3} \mathcal{A} \wedge \star \mathcal{A} - \frac{2\sigma_2}{3} \mathcal{V} \wedge \star \mathcal{A} + a_1 {}^{(1)}T^\alpha \wedge \star {}^{(1)}T_\alpha \left. \right] \\
& - \frac{1}{2\varrho} \left[\left(\frac{w_6}{12} R^2 - \frac{w_3}{12} X^2 + \frac{\mu_3}{12} R X \right) \eta + w_4 {}^{(4)}R^{\alpha\beta} \wedge \star {}^{(4)}R_{\alpha\beta} \right. \\
& \left. + {}^{(2)}R^{\alpha\beta} \wedge (w_2 \star {}^{(2)}R_{\alpha\beta} + \mu_2 {}^{(4)}R_{\alpha\beta}) + {}^{(5)}R^{\alpha\beta} \wedge (w_5 \star {}^{(5)}R_{\alpha\beta} + \mu_4 {}^{(5)}R_{\alpha\beta}) \right]. \quad (44)
\end{aligned}$$

The first two lines represent weak gravity with the gravitational constant κ , the last two lines strong gravity with the dimensionless coupling constant ϱ . The parity odd pieces are those with the constants $b_0, \sigma_2; \mu_2, \mu_3, \mu_4$. In a Riemann space (where $X = 0$), only two terms of the first line and likewise two terms in the third line survive. All these 4 terms are parity even, that is, only torsion brings in parity odd pieces into the gravitational Lagrangian.

Acknowledgments

This work was supported by the German-Israeli Foundation for Scientific Research and Development (GIF), Research Grant No. 1078-107.14/2009.

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